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# Embedding Theorems in Shape Theory (Shapeと無限次元多様体)

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# Embedding theorems in shape theory

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Let  $X$  be a compactum ( a compact metric space ) and let  $E^n$  be an  $n$ -dimensional Euclidean space. Borsuk raised several problems concerning to embed  $X$  into  $E^n$  up to shape ( see [ 1 ] and [ 2 ] ). Partial answers of them were given by Trybulec and many other persons. In section ( 1 ) we shall trace their works again.

The second topic is the Chapman's complementary theorem. After [ 3 ] several attempts to modify it have been done. We shall recall them in section ( 2 ).

## § 1

Borsuk defined the index  $e(X)$  for every compactum  $X$ . But the trouble is that, between [ 1 ] and [ 2 ], definitions of  $e(X)$  are different. So in this note, we use two simbols  $e_1(X)$ ,  $e_2(X)$  if we want to distinguish them.

In [ 1 ],  $e_1(X)$  is defined by

$$e_1(X) = \min ( k \mid E^k \supset \mathfrak{A} Y ; \text{Sh}(X) = \text{Sh}(Y) ),$$

and in [ 2 ],  $e_2(X)$  is defined by

$$e_2(X) = \min ( k \mid E^k \supset \mathfrak{A} Y ; \text{Sh}(X) \leq \text{Sh}(Y) ).$$

Borsuk's problems are followings :

- ( 1 ) To find a pure definition of  $e(X)$ .
- ( 2 )  $Sh(X) \leq Sh(Y) \Rightarrow e_1(X) \leq e_1(Y)$  ?
- ( 3 ) Dose there exist for every  $n$  a compactum  $X$  such that  $Fd(X) = n$  and  $e(X) = 2n+1$  ?
- ( 4 ) Is it true that for every  $n$ -dimensional movable continuum  $X$  the number  $e(X) \leq 2n$  ?

Borsuk pointed out that for  $n = 1$  Problem ( 3 ) has a positive answer. A solenoid is its example ( see [ 2 ] ).

Trybulec [ 15 ] proved the following theorem. It gives an affirmative answer of Problem ( 4 ) for the case  $n = 1$ .

Theorem 1 ( [ 15 ] Th.3.6 ). The shape of every movable curve  $X$  is plane.

Recentry Ivanšić and Husch investigated them. Their works are essentially based on the embedding theorem up to simple homotopy which was proved by Stallings.

Theorem 2 ( [ 14 ] ), If  $K$  is a polyhedron of dimension  $k$ ,  $M$  a manifold of dimension  $m$ , and  $f: K \rightarrow M$  a map which is  $(2k-m+1)$ -connected, then there is a procedure which, when  $k \leq m-3$ , yields a  $k$ -dimensional subpolyhedron  $K_1 \subset M$  and a simple homotopy equivalence  $K \rightarrow K_1$  for which the diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ & \searrow & \nearrow \\ & K_1 & \end{array}$$

is consistent up to homotopy.

Their theorems are following. All of them are pointed case.

Theorem 3 ( [ 10 ] Cor.3 ). Every continuum  $X$  which is pointed 1-movable and  $\text{Fd}(X, x) = n \geq 3$  can be embedded up to pointed shape in  $E^{2n}$ .

A pointed space  $X$  is said to be  $r$ -shape connected provided  $X$  is connected and has trivial homotopy pro-groups for  $1 \leq k \leq r$ . A system map  $\{f_i\} : \{X_i\} \rightarrow \{Y_i\}$  is said to be shape  $r$ -connected if it induces an isomorphism of homotopy pro-groups of  $\{X_i\}$  and  $\{Y_i\}$ , denoted by  $\pi_j(\underline{X})$  and  $\pi_j(\underline{Y})$  for each  $1 \leq j < r$  and epimorphism for  $j = r$  in the category of pro-groups.

A pointed compactum  $X$  has shape finite  $r$ -skeleton ( $r \geq 1$ ) if there exists a finite connected pointed CW-complex  $K$  and a tower of pointed CW-complexes  $\{X_i\}$  such that  $X = \varprojlim \{X_i\}$  and a system map  $\{f_i\} : \{K\} \rightarrow \{X_i\}$  such that  $\{f_i\}$  is shape  $r$ -connected.

Theorem 4 ( [ 10 ] Th.5 ). If  $X$  is a pointed compactum,  $\text{Fd}(X, x) = n$ , which is  $r$ -shape connected,  $n - r \geq 2$ , then  $(X, x)$  can be embedded up to shape into  $E^{2n-r+1}$ .

Theorem 5 ( [ 10 ] Th.6 ). Let  $X$  be a pointed compactum which is pointed shape dominated by a polyhedron and let  $\text{Fd}(X, x) = n \geq 3$ . If  $(X, x)$  has trivial shape groups for  $1 \leq i \leq r$ ,  $n - r \geq 3$ , then  $(X, x)$  can be embedded up to shape into  $E^{2n-r}$ .

Theorem 6 ( [ 8 ] Th.7 ). Let  $M$  be a PL-manifold of dimension  $q$  and let  $X$  be a continuum which has fundamental dimension  $n$ ,  $q - n \geq 3$ , has stable pro- $\pi_1$  which is pro-isomorphic

to a finitely presented group and has a shape finite  $(2n-q+1)$ -skeleton. If there exists a shape map  $\{f_i\} : X \rightarrow M$  which is shape  $(2n-q+1)$ -connected, then there exists a compactum  $Z \subseteq M$  such that  $\text{Sh}(X) = \text{Sh}(Z)$ .

Theorem 7 ( [ 8 ] Th.11 ). Let  $Y \subseteq E^q$  be an  $n$ -dimensional continuum which has stable  $\text{pro-}\pi_1$  pro-isomorphic to a finitely presented group. Let  $X$  be a continuum such that fundamental dimension of  $X \leq n$ ,  $X$  has a shape finite  $(2n-q+1)$ -skelton and  $\text{Sh}(X) \leq \text{Sh}(Y)$ . If  $3n < 2q-2$  and  $n \geq 3$ , then there exists a compactum  $Z \subseteq E^q$  such that  $\text{Sh}(Z) = \text{Sh}(X)$ .

Theorem 8 ( [ 8 ] Cor.12 ). Let  $Y \subseteq E^q$ ,  $q \geq 5$ , be an  $n$ -dimensional continuum which satisfies cellularity criterion ( see section 2 ). If  $X$  satisfies the same conditions as in Theorem 7, then  $X$  can be embedded up to shape into  $E^q$ .

Theorem 9 ( [ 9 ] Th.1 ). Let  $X$  be a continuum of fundamental dimension  $\text{Fd}(X) = k \geq 3$  which is pointed  $(2k-q+1)$ -movable. If there exists a shape  $(2k-q+1)$ -connected shape map  $\underline{f} : X \rightarrow M$  of  $X$  into a  $q$ -dimensional PL-manifold which is either closed or is open and dominated by a finite complex and if  $q-k \geq 3$ , then there exists a  $k$ -dimensional continuum  $Y \subseteq M$  and a shape equivalence  $\underline{g} : X \rightarrow Y$  such that  $\underline{i}\underline{g} = \underline{f}$  where  $\underline{i}$  is the shape map induced by the inclusion  $Y \hookrightarrow M$ .

Theorem 10 ( [ 9 ] Cor.2 ). Let  $X$  be a shape  $r$ -connected continuum of fundamental dimension  $k \geq 3$  which is  $(r+1)$ -pointed movable, then there exists a continuum  $Y \subseteq E^{2k-r}$  which has the

same shape as  $X$ .

They showed that in Theorem 9 the conditions of movability and shape connectivity are essential. In section 4 of [ 9 ], counter examples are given. But the condition of  $(2k-q+1)$ -movability can be replaced with  $S^{2k-q+1}$ -movability.

For Problem ( 3 ), Duvall and Husch constructed such spaces for the case of  $n = 2^k$  ( $k > 1$ ) ( see [ 6 ] ).

## § 2

Before discussing complementary theorems, we must see some definitions of nice embeddings. Let  $X$  be a compactum in a space  $M$ .

- ( 1 )  $X$  is a  $Z$ -set in  $M$  if for every nonempty homotopically trivial open set  $U$  in  $M$ ,  $U \setminus X$  is nonempty and homotopically trivial.
- ( 2 )  $X$  is a  $Z_k$ -set ( $k$  an integer  $\geq 0$ ) in  $M$  if for every nonempty  $k$ -connected open set  $U$  in  $M$ ,  $U \setminus X$  is nonempty and  $k$ -connected.
- ( 3 )  $X$  is a strong  $Z_k$ -set ( $k \geq 0$ ) in  $E^n$  if for each compact subpolyhedron  $P$  of  $E^n$  having dimension  $\leq k+1$  and each  $\epsilon > 0$ , there is an  $\epsilon$ -push  $h$  of  $(E^n, X)$  such that  $h(X) \cap P = \emptyset$ .
- ( 4 )  $X$  satisfies the cellularity criterion ( CC ) if given a neighborhood  $U$  of  $X$ , there is a neighborhood  $V \subset U$  of  $X$  such that every loop in  $V \setminus X$  is null-homotopic in  $U \setminus X$ .
- ( 5 )  $X$  is globally 1- $\text{alg}$  in  $M$  if given a neighborhood

$U$  of  $X$ , there is a neighborhood  $V \subset U$  of  $X$  such that every loop in  $V \setminus X$  which is null-homologous in  $V \setminus X$  is null-homotopic in  $U \setminus X$ .

( 6 )  $X$  satisfies the small loops condition ( SLC ) if for any neighborhood  $U$  of  $X$  there is a neighborhood  $V \subset U$  of  $X$  and an  $\epsilon > 0$  such that each loop in  $V \setminus X$  of diameter less than  $\epsilon$  is null-homotopic in  $U \setminus X$ .

( 7 )  $X$  satisfies the inessential loops condition ( ILC ) if for every neighborhood  $U$  of  $X$  there is a neighborhood  $V \subset U$  of  $X$  such that each loop in  $V \setminus X$  which is null-homotopic in  $V$  is also null-homotopic in  $U \setminus X$ .

Chapman's complementary theorem is following.

Theorem 11 ( [ 3 ] Th.2 ). If  $X$  and  $Y$  are compacta  $Z$ -embedded in the Hilbert cube  $Q$ , then  $X$  and  $Y$  have the same shape iff  $Q \setminus X$  and  $Q \setminus Y$  are homeomorphic.

In this theorem, if we replace  $Q$  with  $E^n$  or  $S^n$  what kinds of conditions are needed ? Chapman showed the next theorem, and Geoghegan and Summerhill improved it.

Theorem 12 ( [ 4 ] Th.1 ). Let  $X, Y$  be compacta such that  $\dim X, \dim Y \leq m$ .

( a ) For any integer  $n \geq 2m+2$  there is copies  $X', Y' \subset E^n$  ( of  $X, Y$  respectively ) such that if  $Sh(X) = Sh(Y)$ , then  $E^n \setminus X'$  and  $E^n \setminus Y'$  are homeomorphic.

( b ) For any integer  $n \geq 3m+3$  there is copies  $X', Y' \subset E^n$  ( of

$X, Y$  respectively ) such that if  $E^n \setminus X'$  and  $E^n \setminus Y'$  are homeomorphic, then  $Sh(X) = Sh(Y)$ .

Theorem 13 ( [ 7 ] Th.1.1 ). Let  $X$  and  $Y$  be nonempty compact strong  $Z_{n-k-2}$ -sets in  $E^n$  (  $k \geq 0$ ,  $n \geq 2k+2$  ). Then  $X$  and  $Y$  have the same shape iff  $E^n \setminus X$  and  $E^n \setminus Y$  are homeomorphic.

Proposition ( [ 7 ] Prop.1.3 ). Every compactum of dimension  $\leq k$  can be embedded in  $E^n$  (  $n \geq 2k+1$  ) as a strong  $Z_{n-k-2}$ -set.

On the other hand, Venema proved the following theorems.

Theorem 14 ( [ 17 ] Th.1 ). Let  $X$  and  $Y$  be compacta in  $E^n$ ,  $n \geq 5$ , satisfying ILC and having shape dimension in the trivial range with respect to  $n$  (  $2Fd(X)+2 \leq n$ ,  $2Fd(Y)+2 \leq n$  ), then  $E^n \setminus X$  and  $E^n \setminus Y$  are homeomorphic iff  $Sh(X) = Sh(Y)$ .

Theorem 15 ( [ 17 ] Th.2 ). Let  $X$  and  $Y$  be globally 1-alg compacta in  $E^n$ ,  $n \geq 5$ , and let  $A, B$  be compact connected abelian topological groups with  $2dim(A)+2 \leq n$ . If  $Sh(X) = Sh(A)$  and  $Sh(Y) = Sh(B)$  then the following are equivalent:

- ( a )  $E^n \setminus X$  and  $E^n \setminus Y$  are homeomorphic,
- ( b )  $Sh(X) = Sh(Y)$ , and
- ( c )  $A$  and  $B$  are topologically isomorphic.

In the case that  $X$  and  $Y$  are compacta in  $S^n$ , Coram and Duvall proved the following theorem.



Theorem 16 ( [ 5 ] Th.3.3 ). Let  $X, Y$  be  $S^k$ -like continua in  $S^n$  such that  $X$  and  $Y$  satisfy SLC,  $1 \leq k \leq n-4$ . Then  $S^n \setminus X$  and  $S^n \setminus Y$  are homeomorphic iff  $\text{Sh}(X) = \text{Sh}(Y)$ .

More generally, there are Rushing and his students' works.

Theorem 17 ( [ 13 ] Th.1 ). Let  $X \subset S^n$ ,  $n \geq 5$ , be compact. Then, for  $k \neq 1$   $\text{Sh}(X) = \text{Sh}(S^k)$  is equivalent to  $S^n \setminus X \cong S^n \setminus S^k$  if  $X$  is globally 1-alg ( and if  $S^n \setminus X$  has the homotopy type of  $S^1$  when  $k = n-2$  ).

Theorem 18 ( [ 16 ] Th.1 ). Let  $X \subset S^n$ ,  $n \geq 5$ , be a globally 1-alg compactum, and let  $S_p^k$  be the  $(k-1)$ -fold suspension of  $S_p^1$  ( a solenoid ). Then  $S^n \setminus X \cong S^n \setminus S_p^k$ ,  $k \neq 1$ , if and only if  $\text{Sh}(X) = \text{Sh}(S_p^k)$  and in case  $k = n-2$ ,  $\pi_1(S^n \setminus X)$  is abelian and  $\pi_i(S^n \setminus X) = 0$ ,  $i \geq 2$ .

Theorem 19 ( [ 11 ] Th.2 ). Let  $X, Y$  be globally 1-alg continua in  $S^n$ ,  $n \geq 5$ , having the shape of finite complexes  $K, L$  ( respectively ) in trivial range such that  $\pi_1(K), \pi_1(L)$  are abelian. If either  $\pi_1(K) = \pi_1(L) = 0$  or  $\pi_2(K) = \pi_2(L) = 0$ , then  $\text{Sh}(X) = \text{Sh}(Y)$  iff  $S^n \setminus X \cong S^n \setminus Y$ .

Theorem 20 ( [ 12 ] Th.2 ). Let  $X_1$  and  $X_2$  be globally 1-alg continua in  $S^n$  ( $n \geq 6$ ) having the shape of codimension 3, closed,  $0 < (2m_i - n + 1)$ -connected topological manifolds  $M_i^{m_i}$ ,  $i = 1, 2$  ( respectively ). Then,  $S^n \setminus X_1 \cong S^n \setminus X_2$  iff  $\text{Sh}(X_1) = \text{Sh}(X_2)$ .

## References

- [1]. K.Borsuk, On several problems in the theory of shape.  
Studies in topology, Academic Press 1975, 67-79.
- [2]. \_\_\_\_\_, Theory of shape. Math. Monographs, Vol.59.  
Warsaw 1975.
- [3]. T.A.Chapman, On some application of infinite-dimensional  
manifolds to the theory of shape. Fund. Math. 76  
(1972), 181-193.
- [4]. \_\_\_\_\_, Shape of finite-dimensional compacta.  
Fund. Math. 76 (1972), 261-276.
- [5]. D.Coram & P.F.Duvall, Jr, Neighborhood of sphere-like  
continua. Gen. Top. and its Appl. 6 (1976) 191-198.
- [6]. P.F.Duvall & L.S.Hush<sup>c</sup>, A continuum of dimension  $n$  which  
does not embed up to shape in  $2n$ -space.(to appear)
- [7]. R.Geoghegan & R.R.Summerhill, Concerning the shape of  
finite dimensional compacta. Trans.A.M.S. 179 (1973)  
281-292.
- [8]. L.S.Hush<sup>c</sup> & I.Ivanšić, Shape domination and embedding up  
to shape. (submitted to Compositio Math.)
- [9]. \_\_\_\_\_ & \_\_\_\_\_, Embeddings and concordances of  
embeddings up to shape. (submitted to Acta Math.)
- [10]. I.Ivanšić, Embedding compacta up to shape. Bull. Acad.  
Polon. Sci. 25 (1977) 471-475.
- [11]. V.T.Liem, Certain continua in  $S^n$  of the same shape have  
homeomorphic complement. Trans.A.M.S. 218 (1976)  
207-217.

- [12]. V.T.Liem, Certain continua in  $S^n$  with homeomorphic complements have the same shape. Fund. Math. 97 (1977) 221-228.
- [13]. T.B.Rushing, The compacta  $X$  in  $S^n$  for which  $\text{Sh}(X) = \text{Sh}(S^k)$  is equivalent to  $S^n \setminus X \approx S^n \setminus S^k$ . Fund. Math. 97 (1977) 1-8.
- [14]. J.Stallings, The embedding of homotopy types into manifolds. Mimeo note, Princeton University 1965.
- [15]. A.Trybulec, On shape of movable curves. Bull. Acad. Polon. Sci. 21 (1973) 727-733.
- [16]. G.A.Venema, Weak flatness for shape classes of sphere-like continua. Gen. Top. and its Appl. 7 (1977) 303-319.
- [17]. \_\_\_\_\_, Embeddings of compacta with shape dimensional in the trivial range. Proc. A.M.S. 55 (1976) 443-448.